Visualising Mathematical Knowledge and Understanding

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***Abstract*—Contemporary mathematics education attempts to instil within learners the conceptualization of mathematics as a highly organized and inter-connected set of ideas. To support this, a means to graphically represent this organization of ideas is presented which reflects the cognitive mechanisms that shape a learner's understanding. This organisation of information may then be analysed, with the view to informing the design of mathematics instruction in face-to-face and/or computer-mediated learning environments. However, this analysis requires significant work to develop both theory and practice.**

***Index Terms*—mathematics, reflective abstraction, graphical representation**

I. INTRODUCTION

Contemporary mathematics education promotes the domain of mathematics to be a highly organized structure of ideas, of which learners develop a rich understanding. This paper reports on an on-going research program that is developing a novel graphical language for creating graphical representations of this structure. The graphical representations reflect the cognitive processes by which learner's make sense of the domain. Included in this paper is a summary of the constructs of the graphical language and an example of its application to the domain of early-number mathematics. It is proposed that the graphical representations may inform the design of instruction. However, this requires significant future work to develop theory and practice related to the graphical representations manipulation and analysis.

II. BACKGROUND

The graphical language that has been developed in this on-going research program is based upon some key educational theories and practices which are summarised in the following sub-sections.

A. Ontological Considerations

Mathematics knowledge is characterised as a highly organised structure of shared ideas, including the problems encountered and solved using mathematics, the set of concepts which (when selected and applied) form the solutions to the encountered problems, and the set of representations that are used to express both the problems and concepts of mathematics. Contemporary mathematics education may be characterised as learners authentically constructing unique, experience-based understandings of the knowledge that is shared in their mathematical community. This involves the exploration of the connections, both within and between the three different types of mathematical ideas. A learners development of a rich understanding of the organisation of ideas is the goal of many approaches to mathematics instruction e.g., [1]. In practice however, the reform of mathematics education (including the advancement of computer-mediated learning) has been stymied and often criticised for inadequately focussing upon mathematics' highly structured nature [2], [3]. It is against this backdrop that the on-going research presented in this paper has been developing techniques for creating models of the upon which learning environments can flexibly and dynamically respond to the idiosyncratic needs and understandings of each learner.

B. Epistemological Considerations

Two key theories have informed the epistemological framework upon which the proposed graphical language is based: Piaget's notion of reflective abstraction and Popper's conceptualisation of knowledge.

Piaget [4] (and as discussed by Dubinsky [5]) proposed reflective abstraction as the mechanism by which an individual makes accommodations in their conceptual schema. Reflective abstraction is based upon the learner looking back upon past experiences or actions and identifying the similarities and differences between these actions. From this, the learner abstracts ideas that organise their experiences. Piaget identified five specific types of reflective abstraction: (a) Interiorisation, involving the internalisation and then representation of an idea using a more de-contextualised form; (b) Coordination, involving the manipulation of parts to form a more complex whole; (c) Encapsulation, involving the formation of a more abstract, manipulable object that represents either a sub-ordinate relationship or a coalescing of parts; (d) Generalisation, involving the broadening of an idea's applicability; and (e) Reversal, involving the noticing of differences between ideas and the subsequent formation of inverse relationships.

Popper [6] differentiated the in-the-head knowledge unique to each learner from the state-able 'theories', or knowledge, of the community. These two 'worlds' of knowledge form a tension: an individual's actions and idiosyncratic understanding are shaped by the knowledge shared by the community, which is in turn extended or refined through the individual's statement of their in-the-head knowledge. The differentiation of each individual learner's unique understanding from the shared knowledge of the community is a central organising idea of the new theory that has been proposed and demonstrated in this study.

Reflective abstraction influences both the organisation of shared knowledge and the learner's unique understanding. As the learner reflectively abstracts and subsequently states their

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new found theories, they will extend the shared knowledge of the community in ways that correspond to the particular form of reflective abstraction. Similarly, as the learner experiences the extant knowledge shared in the community, the organisation of that knowledge will influence their reflective abstraction. When designing educational experiences, the structure of the shared knowledge should inform the selection of learning activities such that the learner's reflective abstraction and subsequent deepened understanding is effectively scaffolded.

C. Graphical Representations of Knowledge

The creation of visual artefacts to express the organisation of ideas in some domain has been used in education for many years. For example, Novak and his colleagues [7] have proposed concept maps as a means for students to create descriptions of domain knowledge. A concept map is a hierarchical organization of ideas from the most general or abstract to those which are most specific. To capture the relationships between the concepts and thus give structure or organisation to the concept map, linking phrases are used which themselves embody further concepts related to the domain. To use the concept map as a mechanism to assess a learner's understanding requires the teacher to interpret both the concepts and the linking phrases.

The adoption of Piaget's reflective abstraction as the key mechanism by which to explain a learner's conceptual growth has led to an alternate approach for graphically representing knowledge. This alternate approach does not rely upon the use of a second set of domain concepts to create relationships. Instead, reflective abstraction provides the basis of constructs with which to define the organisation of shared knowledge. In a complementary way, the alternate approach also supports the modelling of each learner's experiences with reference to the shared knowledge (i.e., their understanding). The graphical model to describe each individual could then be analysed to inform the design of appropriate experiences which are tailored to the individual's needs.

III. The Graphical Language and Its Application

The graphical language proposed to model shared knowledge has been previously introduced in [8] and was more comprehensively applied to the study of early-number mathematics in [9]. In the following paragraphs, the graphical language is re-introduced using a simple example taken from the comprehensive study. This includes the introduction of the language constructs used to describe and analyse a learner's unique understanding. Then, a brief discussion of the comprehensive application of the language to earlynumber is provided, leading to the identification of future research activities.

Seminal early-number literature (e.g., [10]) has classified worded problems in a variety of ways. The author has reviewed this literature in [9] and identified 22 different types (Type A – Type V), each of which is defined by its set of characterising features. An example of Type A problem is Connie has 5 red marbles and 8 blue marbles. How many marbles does she have altogether? An example of Type G problem is Connie had 5 marbles. Jim gave her 8 more marbles. How many marbles does Connie have altogether? These two problems share common features (both addition with an unknown total), but differ in that the Type G problem has an explicit action (Connie's collection of marbles was added to). These problems could be expressed in a variety of ways, including the concrete modelling using marbles to act out the problem, the use of iconic representations (e.g., counters) to similarly act out the problem, or the expression of the problem using a symbolic number sentence (e.g., 5 + 8 = ?). To solve these problems, several strategies might be used, one of which is the count-all strategy. This requires the person to physically count both set of objects separately, then put the sets together and count the resultant set. This strategy can only be expressed using physical models (e.g., the marbles or counters), since it requires objects to count.

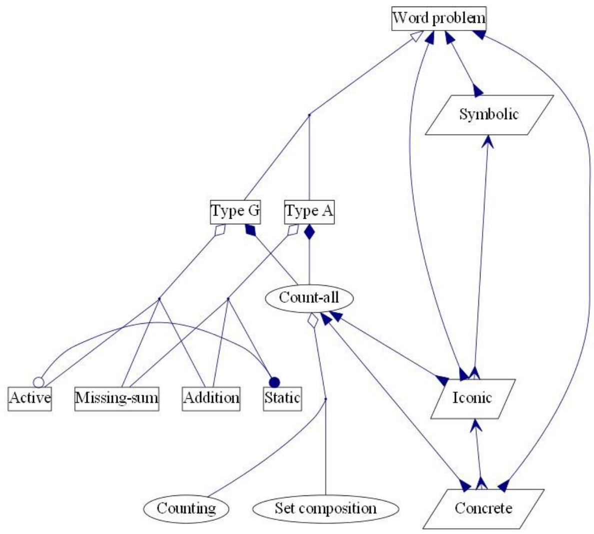

Fig. 1. Genetic decomposition of early-number.

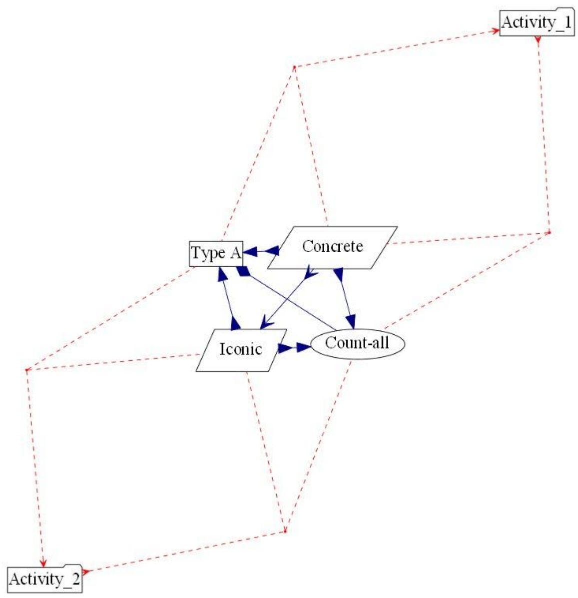

Fig. 2. Developmental trajectory in early-number.

In the graphical language, a genetic decomposition is a network-like representation of shared knowledge. Fig. 1. presents a genetic decomposition that describes the fragment of shared early-number knowledge described in the previous paragraph. In this genetic decomposition, the rectangular icons identify problem knowledge objects, the ellipse icons identify concept knowledge objects, and the trapezoidal icons identify representation knowledge objects. Joining these knowledge objects are lines that describe the relationships, or knowledge objects associations, that define the organisation of the knowledge. The different types of associations are differentiated by the type of terminator used.

In Fig. 1. the similarities between the Type A and Type G problems are described using the inheritance association (indicated by the open triangle terminator) that spans the two problem objects and the more abstract Word Problem object. The two problems types are described in terms of their constituent features: the aggregation association (indicated by the open diamond notation) describes both Type A and Type G to have addition and missing-sum features, but these two problems differ because the Type A problem aggregates the static feature and the Type G problem aggregates the active feature. This difference is highlighted by the inversion association (terminated using open and closed circles) which shows some complementary relationship between the static and active features.

Both the Type A and Type B problems can be solved using the count-all concept and this is indicated by the two solution associations (which are terminated by closed diamonds). The count-all strategy has two constituent parts: counting and set composition. This is described in the genetic decomposition using another aggregation association (again terminated with an open diamond).

Three different types of representation object are identified: concrete, iconic and symbolic. Each of these is progressively more abstract or de-contextualised, which is described using two formalisation associations (each terminated by open arrowheads which point along the path of de-contextualisation). The expression association (terminated by closed arrowheads pointing from representation to problem or concept) is used to describe which problems or concepts can be described using the representations. In this case, word problems can be represented using the concrete, iconic and symbolic representations (requiring three instances of expression association) and the count-all concept can be expressed using only concrete or iconic representations (requiring two instances of expression association).

A learner's understanding of the shared knowledge is based upon their experience. To describe their experience, and thus infer their understanding, the graphical language defines the image construct. This construct is used to describe a single experience in which a problem is solved using one or more concepts, and in which various representations are used to express problem and concept(s). A sequence of images (i.e., a sequence of experiences) can be overlaid upon a genetic decomposition to form what is termed in the language as a developmental trajectory. Such a developmental trajectory, showing two images, is presented in Fig. 2. The first image in the developmental trajectory (labelled Activity 1) describes a Type A problem (expressed concretely) being solved using the count-all strategy (also expressed concretely). The second image in the developmental trajectory (labelled Activity 2) also describes the Type A problem being solved using the count-all strategy, but in this image both the problem and the solution strategy are expressed iconically. Thus, this developmental trajectory describes a learner successfully using the count-all strategy to solve Type A problems with a reflective abstraction-based growth in understanding from highly contextualised representations to more abstract representations (i.e., the reflective abstraction transformation of interiorisation).

The potential of the proposed language was comprehensively demonstrated in [9] using an iterative action-research based methodology. To generate the graphical representations, a tool flow that utilised the graphviz (www.graphviz.org) toolset was used to translate xml-based specifications of genetic decompositions and developmental trajectories into images (png files). In particular, the hierarchical genetic decomposition diagrams were created using the dot tool and the developmental trajectory diagrams were created using the neato tool.

Using this tool flow, seminal literature regarding early-number mathematics was analysed and a set of genetic decompositions were created. These were then synthesised together to form a composite genetic decomposition which represents the shared knowledge of the community. Despite the early-number domain's relative simplicity, the genetic decompositions which were created to describe the domain are quite complicated and difficult to navigate: The composite genetic decomposition comprised of more than 100 knowledge objects and more than 200 organising associations. With the composite genetic decomposition in place, preliminary work in mapping simple developmental trajectories across the early-number domain was then conducted to verify the usefulness of the image construct. Additionally, specific arrangements of knowledge objects that would lead to some advancement of learner understanding were proposed and examples of these were identified in the composite genetic decomposition.

This application of the graphical language to the early-number domain has demonstrated the viability of the language and has set a foundation and direction for future work.

IV. Implications for Future Work

The research program's future work includes the improvement of the tool flow such that genetic decompositions and developmental trajectories can be more easily created, edited, navigated and analysed as needed, such that they become a useful tool to guide the design of mathematics instruction.

Firstly, the toolset could be improved to better support the editing and navigation of genetic decompositions. For example, when considering some focal knowledge object it would be useful to only display those other knowledge objects that are directly associated. Similarly, mechanisms that allow for filtering of the displayed content (e.g., show only concepts, show only inheritance associations) may also be useful. This partial view approach could also be extended such that the view could change as focus objects or associations are selected, requiring the graphic representation to dynamically change as it is traversed.

Secondly and perhaps more importantly, new or improved algorithms for laying out the graphic representations are required. As previously noted, the dot and neato tools were used to create the layouts of the genetic decompositions and developmental trajectories. With particular regard to the genetic decompositions, use of the dot tool was a suitable starting point, since the organisation of knowledge is predominantly hierarchical. However, there are certain aspects of the knowledge associations that are not strictly hierarchical and so not well rendered using the dot tool. For example, the inversion association forces associated nodes onto the same level in the hierarchy, which is not necessarily true or desirable. Similarly, the dot layout forces all components of an aggregation association onto the same level in the hierarchy which is not always necessary to generate a pleasing or meaningful layout. Another limitation of the graphviz toolset is the single-dimension portrayed in the hierarchical output, whereas in fact the three types of knowledge objects (problems, concepts and representations) imply the need to render the hierarchy in three dimensions. This would more clearly and accurately portray the organisation of knowledge and the potential development of understanding. In a related way, use of the neato tool to generate the developmental trajectory images was inadequate because the generated developmental trajectory graphics do not visually describe the generally hierarchical organisation of knowledge. Using the dot tool to create the graphic incorrectly arranged the graphic because the images were interpreted as nodes in the hierarchy rather than being overlaid upon it.

The graphical language and the theoretical framework it embodies have been proposed as a basis upon which to design mathematics instruction, both face to face and computer-mediated, that is more cognizant of the highly structured organisation of mathematics knowledge. In particular, the language allows for the organisation of domain knowledge in terms of the cognitive processes that will lead to the learner's development of deep mathematical understanding. The proposed enhancements will require significant creation of new theory and associated practice that will allow the user to not only create static descriptions of the organisation, but to analyse that structure to identify developmental trajectories that are of significance in relation to the learner's development of deep understanding. The development of computer-mediated learning that uses this new theory to dynamically respond to the learner is a long term goal. In the shorter term, a more achievable goal is to create an environment that allows teachers to construct and navigate genetic decompositions and developmental trajectories such that they are able to make better informed decisions regarding the design of classroom-based mathematics instruction.

References

[1] A. Bell, "Principles for the design of teaching," *Educational Studies in Mathematics*, vol. 24, no. 1, pp. 5-34, 1993.

[2] J. McDonald, S. Yanchar, and R. Osguthorpe, "Learning from programmed instruction: Examining implications for modern instructional technology," *Educational Technology Research and Development*, vol. 52, no. 2, pp. 84-98, 2005.

[3] R. Nason, *et al.*, "Changes in endorsement of an integrated learning system over a period of three years," pretented at 25th Conference of the International Group for the Psychology of Mathematics Education, Utrecht, The Netherlands: PME, 2001.

[4] J. Piaget, "Studies in reflecting abstraction," R. Campbell Trans., 2001, Sussex, England: Psychology Press. (Original work published in 1977)

[5] E. Dubinsky, "Reflective abstraction in advanced mathematical thinking, in Advanced mathematical thinking," D. Tall, Editor 1991, Kluwer Academic Publishers: Dordrecht/Boston/London, pp. 95-123.

[6] K. Popper, Three worlds, in The Tanner Lectures on Human Values1978.

[7] J. Novak and A. Cañas, *The theory underlying concept maps and how to construct and use them*, Technical Report IHMC Cmap Tools, 2008, Florida Institute for Human and Machine Cognition Florida, CA.

[8] D. Nutchey, "A Popperian consilience: Modelling mathematical knowledge and understanding," presented at 34th Annual Conference of the Australasian Mathematics Education Research Group, Alice Springs, Australia, 2011.

[9] D. Nutchey, "Towards a model for the description and analysis of mathematical knowledge and understanding," Queensland University of Technology: Brisbane, Australia, 2011.

[10] K. Fuson, "Research on whole number addition and subtraction, in Handbook of research on mathematics teaching and learning D. Grouws," Editor 1992, Macmillan Publishing Company: New York, pp. 243-275.