The Use of the Lambert Function Method for Analysis of a Multidimensional Control System with Delays and with Structure Having Form of the Star

Irma Ivanovienė and Jonas Rimas

***Abstract*—The mathematical model of the mutual synchronization system, having star form structure and composed of n $(n \in N)$ oscillators, is investigated. The mathematical model of the system is the matrix differential equation with delayed argument. The step responses matrix of the system is obtained applying the Lambert function method. Using obtained step responses of the system the transition processes are investigated.**

***Index Terms*—Synchronization system, differential equations, delayed arguments, Lambert function.**

I. Introduction

The control systems find application in various engineering equipments including the networks of transmitting and distributing of the information. Usually control systems are being investigated applying their mathematical models. More exact analysis of systems demands the use of the more complicated mathematical models. Often the delays of the signals, transfered along the control system, must be included into these models. The delays make the investigation of the model more complicated. Despite the great achievements in the projection and implementation of control systems with delays, the works devoted to analytical investigation of such systems are important.

II. Formulation of the Problem

In the presented work the dynamics of the multidimensional control system with delays and with star form structure is investigated. The mathematical model of this system is the matrix differential equation with delayed argument [1]-[4]

$$\begin{aligned} &x'(t) + B_1 x(t) + B_2 x(t-\tau) = z(t), \\ &x(t) = \phi(t),\ t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $x(t) = \begin{pmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{pmatrix}^T$ is the desired vector function, T denotes the operation of transposition, τ is a constant time delay, $\phi(t)$ is a vector valued initial function, $z(t)$ is a free term (continuous function depending on the initial conditions), κ is a coefficient, B_1 and B_2 are $n \times n$ ($n \in N$) numerical matrices $\left(B_1, B_2 \in R^{n\times n}\right)$,

$$B_1 = \kappa E, \tag{2}$$

$$B_2 = \frac{\kappa}{n-1} B \tag{3}$$

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ n-1 & 0 & 0 & \ddots & \vdots \\ n-1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ n-1 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

($E \in R^{n\times n}$ is the identity matrix, matrix $B \in R^{n\times n}$ outlines the structure of the internal links of the system).

As an example of a control system, described by the equation (1), the mutual synchronization system of the communication network , composed of n oscillators and having star form structure, can be pointed out [3] (Fig. 1).

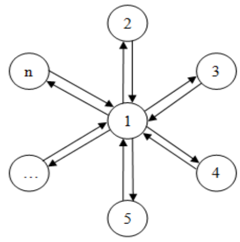

Fig. 1. The scheme of internal links of the system

In this case the symbol $x_i(t)$ in (1) stands for the phase of the i -th oscillator. Taking into account the system's reaction to unit jumps of the phases of oscillations of the oscillators, we shall investigate the transient processes in the synchronization system. For this purpose, firstly, we shall find the step responses matrix of the synchronization system.

III. Step Responses Matrix of the System

The matrix $h(t) = \left(h_{ij}(t)\right)$ we shall call the step responses matrix of the synchronization system. The entry $h_{ij}(t)$ $\left(i, j = \overline{1,n}\right)$ of this matrix is the response of the i -th oscillator oscillation phase to a unit jump in the j -th oscillator oscillation phase. We shall find the matrix $h(t)$.

When the increment of the phase of the j -th oscillator

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takes form of the unit jump the increment of the free term of the equation (1) can be expressed as follows

$$\Delta z(t) = \delta(t) I^{(j)}; \qquad (5)$$

Here $I^{(j)}$ is the matrix-column all elements of which are zeros except the j-th element, which is equal to 1, $\delta(t)$ is the Dirac delta-function. Taking this into account and using (1), we get the following differential equation for step responses $h_{ij}(t)$ $\left(i, j = \overline{1,n}\right)$ of the system:

$$\begin{aligned} &\left(h_j(t)\right)' + B_1 h_j(t) + B_2 h_j(t-\tau) = \delta(t) I^{(j)}, \quad j = \overline{1,n}, \\ &h_j(t) = 0, \ t \in [-\tau, 0]; \end{aligned} \qquad (6)$$

Here $h_j(t) = \left(h_{1j}(t) \ h_{2j}(t) \ \ldots \ h_{nj}(t)\right)^T$ is the j-th column of the step responses matrix $h(t)$, matrices B_1 and B_2 are defined by (2) and (3), respectively.

Using the solution, found on the interval $[0, \tau]$, the differential equation (7) on the interval $[\tau, +\infty)$ can be presented as homogeneous matrix delay differential equation:

$$\begin{aligned} &\left(h_j(t)\right)' + B_1 h_j(t) + B_2 h_j(t-\tau) = 0, \quad j = \overline{1,n}, \\ &h_j(t) = \left(h_{1j}(t) \ h_{2j}(t) \ \ldots \ h_{nj}(t)\right)^T = \phi_j(t), \quad t \in [0, \tau]; \end{aligned} \qquad (7)$$

Here $\phi_j(t)$ is the preshape (initial) vector-function. The entries of the vector-function $\phi_j(t)$ assume the following values:

$$\phi_{ij}(t) = \left(\phi_j(t)\right)_i = \begin{cases} e^{-\kappa t} 1(t), \ \text{if } i = j, \\ 0, \ \text{if } i \neq j; \end{cases} \qquad (8)$$

here $1(t)$ is the Heaviside step function.

Applying the Lambert function method the solution of (7) on the interval $[\tau, +\infty)$ can be expressed as follows:

$$h_j(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k(j) = \lim_{N \to \infty} \sum_{k=-N}^{N} e^{S_k t} C_k(j), \ j = \overline{1,n}, \ t \in [\tau; +\infty]$$

Here $S_k = \frac{1}{\tau} W_k(-B_2 T e^{B_1 \tau}) - B_1$,

$W_k(-B_2 T e^{B_1 \tau})$ is the value of the k-th branch $W_k(H)$ of the matrix Lambert function $W(H)$ at $H = -B_2 T e^{B_1 \tau}$, $C_k(j)$ are the complex valued vectors corresponding to the preshape vector function $\phi_j(t)$ (see (7) and (8)). From (9) follows the approximate expression for $h_j(t)$:

$$h_j(t) = \sum_{k=-N}^{N} e^{S_k t} C_k(j), \ j = \overline{1,n}, \quad t \in [\tau, +\infty); \qquad (10)$$

Here N is sufficiently large natural number.

IV. Step Comparing the Lambert Function Method with the Method of Consequent Integration (Method of 'Steps')

The solution of homogeneous matrix delay differential equation (7) is presented by the infinite functional series (see (9)), which determines the exact solution. In the real calculations we apply the approximate formulas (10), obtained from (9) with finite N ($2N+1$ indicates the number of branches of the Lambert W function, which are used in calculations of the solution).

We shall investigate the rate of convergence of the approximate solution to the exact solution with increasing N. For this purpose we shall apply the exact expression of the step response of the mutual synchronization system with star form structure. This expression we shall find by the method of consequent integration (method of "steps") [5].

The solution of (1), applying the Laplace transform, we present as follows:

$$x(t) \div \sum_{l=1}^{L} (A^{-1} B_2 e^{-pt})^l A^{-1} Z(p), \qquad 0 \le t \le (L+1)\tau;$$

Here $A = pE - B_1 = (p+\kappa)E$, $Z(p) \div z(t)$, $Z(p)$ is the Laplace transform of the vector function $z(t)$ (sign $\div$ links function with its Laplace transform), $L = 0, 1, 2, \ldots$. Taking into account (3), we write

$$x(t) \div \sum_{l=0}^{L} \left(\frac{\kappa}{n-1}\right)^l \frac{1}{(p+\kappa)^{l+1}} e^{-pl\tau} B^l Z(p), \ \ 0 \le t \le (L+1)\tau.$$

Write down the step responses matrix of the system. Using (8), we obtain

$$h(t) = (h_{ij}(t)) \div \sum_{l=0}^{L} \left(\frac{\kappa}{n-1}\right)^l \frac{1}{(p+\kappa)^{l+1}} e^{-pl\tau} B^l, \ \ 0 \le t \le (L+1)\tau.$$

The inverse Laplace transform, applied to the latter expression, gives

$$h_{ij}(t) = \sum_{l=0}^{L} \left(\frac{\kappa}{n-1}\right)^l \{B^l\}_{ij} \frac{(t-l\tau)^l}{l!} e^{-\kappa(t-l\tau)} 1(t-l\tau), \ \ 0 \le t \le (L+1)\tau;$$

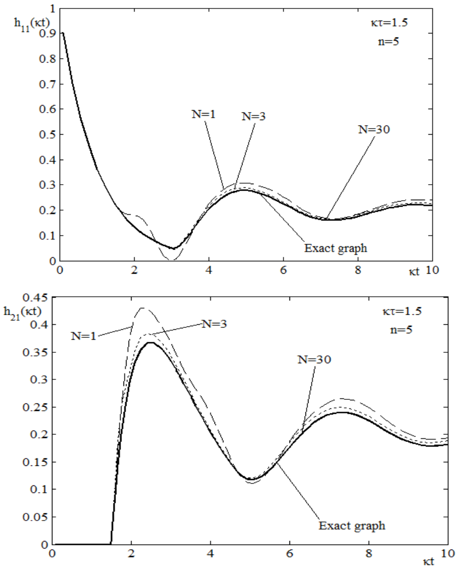
Here $\{B^l\}_{ij}$ is the ij-th element of the matrix B^l, $1(t)$ is the Heaviside step function).

The step response of mutual synchronization system with a structure of a star, computed by the method of consequent integration (the exact method) and by Lambert W function method with different values of N, are presented in the Fig.2. As we see from this figure, increasing N the approximate solution approaches the exact solution.

The maximal relative errors $\delta_{\max}$ obtained for $\kappa t \in [0, +\infty)$ using different values of N are presented in the Table 1, when $n = 5$, and when $n = 15$. As we see from the table for $N = 50$ the maximal relative error is not greater than 0.01 (with increase of N the maximal relative error decreases). Dependence from n of this relative error is insignificant. Such accuracy is sufficient for practical applications.

TABEL I.

N	$n = 5$	1	3	30	50	$n = 15$	1	3	30	50
$\delta_{\max}$		0.6801	0.0805	0.0131	0.0082		0.6820	0.0850	0.0138	0.0086



(a) (b)

Fig. 2. Graphs of the step responses $h_{i1}(\kappa t)$ at different values of N.

V. Results of Calculations

The transients in the synchronization system were investigated applying derived formulas. Some results of calculations are presented in Fig. 3, 4 as graphs of step responses.

For the calculation of the step responses we have applied the approximate formula (10) with $N = 50$ (this means that we have used 101 branches of the Lambert W function in the computations). With such N the relative error is not greater than 0.02 for any κt on the base of the 4-th section. So the graphs of the step responses, presented below, are sufficiently accurate (in the presented figures these graphs practically coincide with the exact ones).

In Fig.3 the graphs of the step response $h_{22}(\kappa\tau)$ are given for different values of product $\kappa\tau$ and for different numbers of oscillators in the synchronization system. From the figure we see that the duration of transients in the synchronization system depends on the magnitude of the product $\kappa\tau$. With increase of $\kappa\tau$ the duration of transients in the system tend to increase. With increase of n the duration of transients in the system changes marginally. The transients get oscillatory features if $\kappa\tau \geq 1.5$. In Fig. 4 the graphs of the step response $h_{11}(\kappa\tau)$ for different values of n are presented. From the figure we see that the dependence on n of the step response $h_{11}(\kappa\tau)$ is not significant.

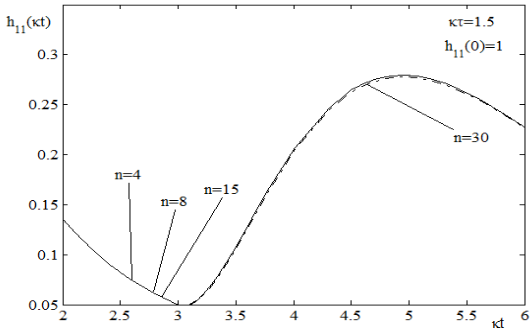
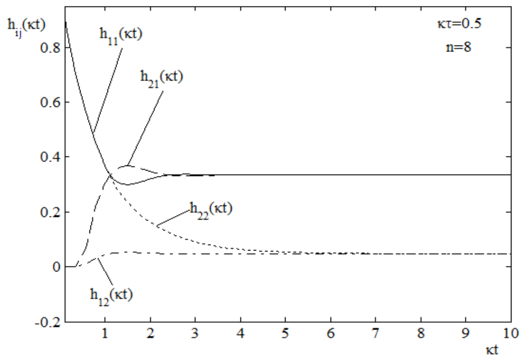
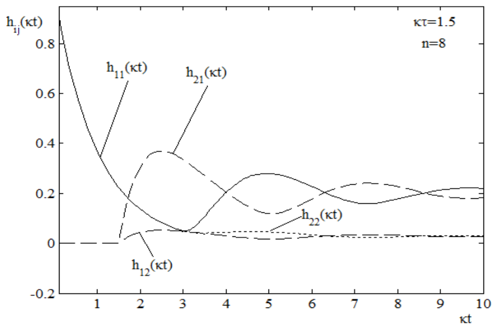


Fig. 4. Graphs of the step responses $h_{11}(\kappa\tau)$.



(a)



(b)

Fig. 5. Graphs of the step responses $h_{ij}(\kappa t)$.

In Fig. 5 the reaction of the oscillations of different oscillators of the system to the unit jump in the phase of the oscillations of the first and second oscillators in the cases $n = 8$ are presented (n is the number of oscillators in the system). As we see from the figure, when the unit jump is given to the phase of the oscillations of the first (the central) oscillator, then the reaction of all oscillators (including the central) is more significant comparing them with the reaction in the case, when the unit jump is given to the not central oscillator

VI. Conclusion

The Lambert W function method is used for computing step responses for the synchronization system. It is shown that using 101 branches of the Lambert W function (taking $N = 50$) in calculations of step responses $h_{ij}(\kappa t)$ the relative error is not greater than 0.01 for any κt and practically does not depend on the number of oscillators in the system.

The Lambert W function method has the advantage in comparison with a method of consequent integration (method of "steps"), as time of calculation of the step response by this method does not depend on delay size, whereas time of calculation of the step response by means of a method of consequent integration is in inverse proportion to the delay size.

The method of research of dynamics, used in the presented work, can also be applied to other control systems, described by the linear matrix differential equations with delayed arguments and with commuting coefficient matrices.

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