Student' Algebraic Thinking Level

Septiani Y. Maudy, Didi S., and Endang M.

***Abstract*—Learning algebra as a bridge to improve mathematical ability in various aspects should not collapse. Many students get no meaning when learning algebra. Students' perceptions of algebra are difficult because there is a variable, therefore students have inherited difficulty. This article aims to share conceptual framework and to analyze research data that lead to algebraic thinking level of students. To be able to help students of various characteristics and ways of thinking, we as educators must know the difficulties they face and at which level the student's algebraic thinking is. So we can anticipate and facilitate them so they can bridge each level well and can help students improve their algebraic thinking.**

***Index Terms*—Algebraic thinking, conceptual framework, learning obstacle.**

I. Introduction

Algebra learning is synonymous with the existence of variables that present a challenge for the students, and of course for the teacher. As one of the critical aspects of mathematics, algebra presents many benefits to life that are not easy to explain to students. Students assume that learning 'x and y' is not real. Although it has been assisted with the context of learning that is close to the life of students, frequently from those who have difficulty. Compared with teaching symbolic and procedural manipulation to students, it should be preceded by an understanding of the whole concept. This understanding also needs to be accompanied by meaningfulness for students.

In solving contextual problems, students need to translate the context in mathematical language or mathematical model, in this case, in particular the linear equations in one variable shaped $x \pm b = c$ and $ax \pm b = c$. In the completion, the students still consider it as an arithmetic equation, but this can already be made into algebraic equations. That is, students still cannot think algebraically, they still think arithmetically. The researcher wants them to start to think algebraically. They find it difficult to make it into algebraic equations because it is not accustomed in the daily learning in the classroom. Students must learn gradually, the linear equation in one variable shaped $x \pm b = c$ can be said is still simple, but to be able to make a mathematical model shaped $ax \pm b = cx \pm d$ of course students must be able first in making mathematical model shaped $x \pm b = c$ [1].

It is not just related to translating contexts into algebraic equations. By the time the students have succeeded in making algebraic equation modelling, in this case the linear equation in one variable shaped $x \pm b = c$ and $ax \pm b = c$, the students still have difficulty in solving it. It is suspected that they memorized the procedures that teachers taught at school. So when they forget the procedure, they made many mistakes in finishing it. Students need to know that there are many ways that can be used to solve linear equations in one variable. Researchers believe that Indonesian students are able to find many ways, but we need to give time and opportunity for students to think, obviously, with the help of a teacher and hints in form of questions that encourage students to be in the desired direction. Students must know the reason of why in the end, of all the ways that exist, the way used often is A or B, not C, D, or others. So that students have meaning in learning. When faced with similar problems, students can choose the method they use based on their previous experience, why students use A or B means that there is a reason. Each step needs to be reasoned, this will avoid the students from just being a memorizer of the procedure because by finding it, they will have its meaning.

The researcher conducted research on learning obstacle students in learning linear equations in one variable (Maudy, 2015). The researcher provides simple problems like the following on the number 5 of learning obstacle test.

If $3 + 2x + x = 24$, specify the value of x.

The following (see Fig. 1) are the results of the answers of students of grade VIII, IX, X, XI who experienced errors.

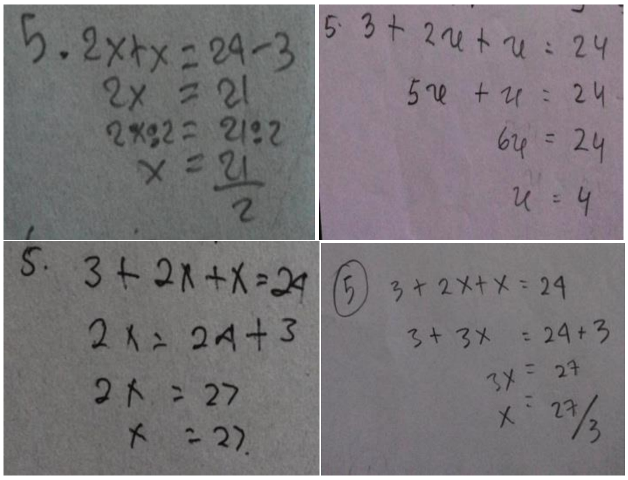

Fig. 1. Student' leaning obstacle.

The errors indicate that students cannot solve linear equations in one variable in the form of $x \pm b = c$ and $ax \pm b = c$, so the researcher claims that there are learning obstacles related to solving linear equations in one variable shaped $x \pm b = c$ and $ax \pm b = c$.

Initially, the researcher doubted whether such a simple problem would be able to see the learning obstacle of students. The problem was categorized as easy, not contextual, but it is not without purpose the researcher gave the problem. The researcher predicted that if students have no meaning to the variables and also learn linear equations in

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one variable just in a rote procedure, students will experience many mistakes while doing it. Apparently the researcher's prediction was true, and not only junior high school students but high school students also experienced many errors. While waiting for junior high school students to work on the problem, the researcher went around. The researcher interviewed a student who had worked on the problem. The following is questions and answers of the researcher and the student.

Researcher: "How were the questions?"

Respondent: "They were hard. Confusing. Since there are x's."

Researcher: "Oh, do you know what x is?"

Respondent: "x is a variable, Ma'am."

Researcher: "Yes it is a variable, but what is a variable?"

Respondent: "No idea, Ma'am."

Researcher: "You have already studied the linear equations in one variable before, aren't you?"

Respondent: "The variable ... hmm ... no idea, Ma'am (laugh)."

Researcher: "Well, it's okay. Can you solve number one?"

Respondent: "I was able to solve number one, Ma'am"

Researcher: "How about number two?"

Respondent: "Number two… hmm… it was complicated since there are x's."

Researcher: "Well no problem. How about question number 6?"

Respondent: "Hmm… oh it is errr…. Yes I can, Ma'am. I tried one by one from one, and it works on the 16th time. It is long, isn't it? Complicated. One variable."

Researcher: "Yes, but that can actually be done with a linear equation in one variable. Can you do that?"

Respondent: "I didn't think about that. I don't know the method. I forgot it, Ma'am (laugh)."

Researcher: "Well, it's okay then. After this, you need to learn the linear equation in one variable again, okay? Thank you."

Respondent: "Yes, Ma'am. You are welcome." (Transcript of interview video) [2].

Students learn algebra without the meaning of the variables, they just manipulate the symbols and imitate the book or teacher's exemplified procedure without conceptualizing. Learning algebra is sure challenging for students, especially in grade VII is the first time they are faced with the unknown. And as time goes by, they still carry the difficulties to the next school level. Some of them are capable of passing through it, but are dry of the meaning and power of the concept. Here, it is the teacher's duty to help them. Each student has different characteristics, where each student passes different difficulties and has different ways of thinking. Teachers need to facilitate these different ways of thinking as well as different levels of thinking, especially on algebra.

Researcher gave the following problems in learning at the time of the research [2].

Masha and Nobita have instagram accounts. There are 9 photos in Masha's instagram and there are 3 photos in Nobita's. Starting next week (first week), Masha will regularly post 2 photos per week and Nobita will regularly post 3 photos per week. How many weeks will it takes to make the number of photos on Masha's instagram the same as Nobita's? Can you define it in several ways?

As the students completed the problem on the worksheets, the researcher went around to check the student's answers, check the diversity of student responses, and help students who are experiencing difficulties. The researcher asked 4 students to write down the answers and present them in front of the class, because the four students worked differently. To reach the intended conclusion, the researcher called students structurally from the simplest to the most complex. Therefore the researcher called the student2 first.

Fig. 2. Student2 presented her answer.

Student2 did it by counting (see Fig. 2), Student 2 had not changed it into algebraic equations.

Fig. 3. Student3 presented her answer.

Student3 (see Fig. 3) and the next called friends have done it by turning the story into algebraic equations. However, the way they solve algebraic equations varies.

Both students have different levels of algebraic thinking when they are in the same level of grade VII. For each student, the teacher needs to be a facilitator who can facilitate different ways of thinking.

How can we help students, if we do not know what the difficulties faced by each student. Each student has different characteristics and different ways of thinking. As teachers, we should be able to anticipate the diversity of student characteristics. We should be aware of the level of algebraic thinking of students as early as possible, so that we can help them to bridge each level well. Students can increase their level, if they are aware of their positions and try to bridge to the next level of course with the help of teachers and friends. So that students can increase their algebraic thinking level with meaning. Based on the explanation above, the research question is “how are the stages of students’ algebraic thinking level?”

II. Theoretical Framework

Thinking algebraically can also be understood as an approach to quantitative situations, as Kieran claims as follows: “Algebraic thinking can be interpreted as an approach to quantitative situations that emphasizes the general relational aspects with tools that are not necessarily letter- symbolic, but which can ultimately be used as cognitive support for introducing and for sustaining the more traditional discourse of school algebra” [3].

Kaput suggests that the domain of algebra consists of both particular thinking practices and content strands. In particular, he contends that algebraic thinking includes (a) making and communicating generalizations in increasingly formal and conventional symbol systems and (b) reasoning with symbolic forms. [4] He additionally contends that these practices occur across three content strands:

1) Algebra as the study of structures and systems abstracted from computations and relations, incorporating those emerging in arithmetic (algebra as generalized arithmetic) and quantitative reasoning.
2) Algebra as the study of functions, relations, and joint variation.
3) Algebra as the application of a cluster of modeling languages both inside and outside of mathematics [4].

From the analysis and drawing from the language of learning progressions research, there are five big ideas which was identified and represented in Kaput’s content strands and around which a lot of early algebra research has matured [5]-[7]. These enormous thoughts, which offer huge opportunities for engaging in the core algebraic thinking practices of generalizing, representing, justifying, and reasoning with mathematical relationships, involve a) equivalence, expressions, equations, and inequalities; b) generalized arithmetic; c) functional thinking; d) variable; and e) proportional reasoning [4], [5].

The huge thought of equivalence, expressions, equations, and inequalities involve building up a relational understanding of the equal sign, representing and reasoning with expressions and equations in their symbolic form, and describing relationships between and among generalized quantities that could conceivably be equivalent. We take generalized arithmetic to include generalizing arithmetic relationships, including fundamental properties of number and operation (e.g., the Commutative Property of Addition), and reasoning about the structure of arithmetic expressions as opposed to their computational value. Functional thinking includes generalizing relationships between covarying quantities and representing and reasoning with those relationships through natural language, algebraic (symbolic) notation, tables, and graphs. Variable alludes to symbolic notation as a linguistic tool for representing mathematical ideas in brief ways and involves the different roles variable plays in various mathematical contexts [5]. Finally, in the context of our work, proportional reasoning alludes to opportunities for reasoning algebraically about two generalized quantities related in such a way that the ratio of one quantity to the other is invariant.

Radford formulates the characteristics of algebraic thinking as follows [8].

1) One deals with a sense of indeterminacy that is proper to basic algebraic objects such as unknown, variables and parameters (someone deals with something uncertain according to the basic object of algebra as unknown, variables, and parameters).
2) Indeterminate objects are handled analytically (objects that must be handled analytically).
3) The peculiar symbolic mode that it has to designate its objects (the use of certain symbols to design the object).

Kriegler emphasizes that there are two parts in algebraic thinking, which are: 1) the development of mathematical thinking tools and 2) the study of the basic idea of algebra. The mathematical thinking tools by Kriegler comprises of three categories: tools for problem-solving skills, representational skills, and quantitative reasoning abilities. While the basic ideas of algebra in question are algebra as a form of arithmetic generalization, algebra as the language of mathematics, and algebra as a tool for the function and modelling mathematics [9]. Lins mentions the characteristics of algebraic thinking as follows. To think algebraically is:

1) To think arithmatically, which means modelling in numbers;
2) To think internally, which means reference only to the operations and equality relation, in other words solutions in the boundaries of the semantic field of numbers and arithmatical operations;
3) To think analitically, which means what is unknown has to be treated as known [10].

TABLE I: Types in Thinking Algebraically

Thinking Algebraically	Explanation
Generalization	Generalization is the process of finding patterns or shapes, which begins with the pattern identified from the given object. Each functional relationship is also a pattern.
Abstraction	Abstraction is a process to extract mathematical objects and relationships based on generalization. Symbols are used in abstractions.
Analytical Thinking	Analytical thinking is the process of applying the inverse operation used under the conditions of the problem in order to find the conditions required for completion.
Dynamic Thinking	Dynamic thinking is thinking by involving variables as objects that can be changed.
Modelling	Modeling is a process for representing complex situations using mathematical expressions, to investigate situations with models, and to describe the relationships of an activity. This representation can use an equation and solve the equation.
Organization	Organizing provides a variety of thinking combinations to find all independent variables, which are important in various solving activities.

Lew defines the thinking types in thinking algebraically. Lew claims that algebra is a way of thinking where the success of algebra is based on six types of mathematical thinking, which are generalization, abstraction, analytical thinking, dynamic thinking, modelling, and organization that will be described in the table as follows [11].

III. Discussion

Ake *et al*. and Godino *et al*. distinguished the algebraization levels of mathematical activity in primary education, including three new levels that characterize secondary mathematics [12], [13]. As a summary, we propose the following six levels of algebraic thinking in primary and secondary education (along with level 0, showing the absence of algebraization):

Level 0: Operations with specific objects using natural, numerical, iconic, gestural languages are done.

Level 1: First encounter with the "generic number", the algebraic structure properties of N and the algebraic equality (equivalence). That is, relational thinking.

Intensive objects, whose generality is explicitly perceived by natural, numerical, iconic or gestural languages, are included. Symbols referring to the perceived intensive objects are used, yet there is no operation with those objects. In structural tasks, relationships and properties of operations are applied and symbolically expressed unknown data might be included.

Level 2: First encounter with the alphanumeric representation of functions and equations and simplifying expressions.

Indetermination or variables conveyed in literal-symbolic language to allude the recognized intensive objects are involved, but they are linked to the spatial or temporal information of the context. In structural tasks the equation is in the form of $Ax \pm B = C$. In functional tasks the generality is recognized, yet there is no operation with variables to get the canonical forms of expression.

Level 3: First encounter with the treatment of unknowns and variables using structural properties (cancellation, replacement, etc.) and the algebraic and functional modeling.

Intensive objects, which are represented literal-symbolically, are generated and operations are done with them; transformations are made in form of symbolic expressions preserving equivalence. Operations are performed on the unknowns to solve equations in the form of $Ax \pm B = Cx \pm D$, and symbolic and decontextualized canonical rules of pattern and function expressions are formulated. In our proposition, level 3 of algebraization assumes the operation with the intensive objects represented symbolically, and in this manner those objects have any contextual connotations.

Level 4: First encounter with the utilization of parameters in functions and variable coefficients, which are second order intensive objects with the expression of families of equations and functions.

The utilization of parameter for expressing equations and function families is indicative of a higher level of algebraic reasoning, regarding the third algebraization level considered by Ake *et al*., which is linked to the processes of "operation with an unknown or variable" [14]. This is a “first encounter” with parameters and coefficient variables involving discrimination of domain and range of the parametric function which is the function assigning a specific function or equation to each value of the parameter. As Ely and Adams state, “A significant conceptual shift must occur in order for students to be comfortable using placeholders in algebraic expressions rather than just numbers” [14].

Level 5: First encounter with the joint treatment of unknowns, variables and parameters, as well as the structure of the solution emerging from the parameter treatment.

We can link a higher level of algebraization to mathematical activity showed when analytical (syntactic) calculations are done in which at least one parameter is involved. Operations with parameter include more prominent semiotic complexity level, since objects emerging from these systems of practices put at stake algebraic objects of the previous level (equations or functions families).

Level 6: First encounter with the study of algebraic structures themselves, their definitions and structural properties.

The introduction of certain algebraic structures (for example, vector spaces or groups) and the study of functional algebra (addition, subtraction, division, multiplication, and composition) begin at high school, applying algebraic objects and processes of higher level of onto-semiotic complexity than those considered at level five. Therefore, it might be useful to characterize the sixth algebraization level that will help us to focus our attention on the particular nature of the involved mathematical activity. High school books include texts and activities correspond to this sixth algebraization level.

In the latest curriculum in Indonesia, the 2013 curriculum, the competencies of high school mathematics in algebraic aspects have been elaborated. Mathematics competencies of junior high school level in the algebraic aspect include using the set, algebraic expressions, relationships and functions, comparisons, social arithmetic, linear equations and linear inequalities in one variable, two-linear equations, straight-line equations, and equations and quadratic functions in solving daily life problems [15].

The mathematics competencies of senior high school level in the algebraic aspect are: using the linear equations and inequalities of one variable of absolute values, system of linear equations in three variables, functions, mathematical logic, mathematical induction, linear program in two variables, matrix, and sequence and series in solving daily life problems High school mathematics is divided into 2 parts: compulsory mathematics and elective mathematics.

In compulsory mathematics are using algebra which includes: linear equations and linear inequalities of absolute values, system of linear equations in three variables, functions (relations and functions, composition of polynomial functions and rational functions, inverse functions), mathematical logic (and questions with quantifier, as well as formal reasoning of inductive, deductive, and deniers to test the validity of arguments, mathematical inductions), linear inequalities in two variables, linear programming, matrices (order, type, operation, transpose, determinant, determinant property, inverse for 3 × 3 matrices), sequence and series (arithmetic, geometry, and indeterminate series).

In elective mathematics, using algebra includes two-sided equation systems (linear-square and square-square), inequality systems in two variables (linear-square and square-square), exponential and logarithmic functions, absolute inequalities, fractional, and irrational, scalar, vectors, vector operations, vector lengths, angles between vectors in two- and three-dimensional spaces, polynomials and operations on polynomials (addition, subtraction, multiplication), division, residual proposition, and polynomial factorization.

IV. Conclusion

Based on the theoretical review, the 2013 curriculum, and the research results of previous researcher, five levels of student algebraic thinking are concluded as follows.

Level 0: Using ways of arithmetic, such as counting and operation with extensive object.

Level 1: Indication of factual generalization.

Level 2: Understanding one unknown and operating it. The generational activities of algebra involve the forming of the expressions and equations that are the intensive objects of algebra. Contextual generalization indicated.

Level 3: Viewing the relationships between variables. The transformational activities includes, for instance, collecting like terms, solving equations, simplifying expressions, working with equivalent expressions and equations, and so on. Viewing the relationships between variables.

Level 4: Using parameters and variable in generational activity. Symbolic generalization indicated.

Level 5: Treatment with parameters.

Level 6: Performing analysis with algebraic structures.

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