Relationship between Discrete Fourier Transformation and Eigenvalue Decomposition

Qun Wan, Li Hong Guo, Ding Wang, Lin Zou, and Ji Hao Yin

***Abstract*—Discrete Fourier transformation (DFT) of sample sequence and eigenvalue decomposition of sample correlation matrix are two of important tools and basic parts in signal and information processing. Since they are used to deal with the same random process, although from different viewpoint, there may be some intrinsic relationship between them. However, they are often introduced, explained and learned independently in the traditional textbooks and courses of signal and information processing. Here, we discuss some intrinsic relationship between the problems formulation of discrete Fourier transformation of sample sequence and eigenvalue decomposition of sample correlation matrix. The results of these lecture notes can help students deepen the understanding of their characteristics on simplicity, optimality and the reason why they are so popular and why we analyze and deal with signal and information processing by using discrete Fourier transformation of sample sequence and eigenvalue decomposition of sample correlation matrix.**

***Index Terms*—Random process, autocorrelation matrix, Discrete Fourier transformation, eigenvalue decomposition.**

I. Introduction

In the course of theory, algorithm and application of signal processing, Discrete Fourier Transformation (DFT) of sample sequence vector and eigenvalue decomposition of sample autocorrelation matrix are two popular tools and play an important role. Since they are used to analyze the same signal or random process, although from different viewpoint, there may be some intrinsic relationship between them. However, they are often introduced, explained and learned independently in the traditional textbooks and courses of signal and information processing. In addition, the students may wonder why $e^{j\omega t}$ is used in Discrete Fourier Transformation of sample sequence vector and why we use normalized vector in eigenvalue decomposition of sample autocorrelation matrix [1].

In this paper, we first introduce the intrinsic relationship behind the definition of Discrete Fourier Transformation of a stochastic process and eigenvalue decomposition of its autocorrelation matrix, from which the students can grasp their related property and review the following knowledge points: the relationship between the variance of a random process and the power spectrum density function, and the relationship between power spectral density function of the input and output stochastic process of a linear time invariant filter [2].

Second, we discuss some intrinsic relationship between the problems formulation and introduced respectively the motivation to analyzing signal and information by using Discrete Fourier Transformation of sample sequence vector and eigenvalue decomposition of sample autocorrelation matrix [3]. It can provide a new perspective for students to understand why we analyze and deal with signal and information processing by using Discrete Fourier transformation of sample sequence and eigenvalue decomposition of sample correlation matrix.

Thirdly, in order to make the conclusions more rigorous, some derivations are provided. As the result, students could have a deeper understanding on the simplicity, optimality and the reason why Discrete Fourier Transformation of sample sequence vector and eigenvalue decomposition of sample autocorrelation matrix are so popular in the field of signal and information processing [4].

This lecture note is organized as follows. Relationship between the definition of Discrete Fourier transform of sample sequence vector and eigenvalue decomposition of sample autocorrelation matrix is introduced in the second section. Section III briefly formulates the problems associated with Discrete Fourier Transformation of sample sequence vector and eigenvalue decomposition of sample autocorrelation matrix. The next section presents the solution to the associated problems. The last section provides a concluding remark to summarize the lecture notes.

II. Relationship between the Definition of Discrete Fourier Transformation and Eigenvalue Decomposition

A. Definition

In the course of signal and information processing, discrete Fourier transformation is usually used to estimate the power spectral density function which is defined as

$$s(\omega)=\sum_{k=-\infty}^{+\infty} r(k)e^{-j\omega k} \tag{1}$$

where

$$r(k)=E\left(x(t)x^{*}(t-k)\right) \tag{2}$$

is the autocorrelation function of random process $x(t)$.

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Given a sample sequence of random process, the power spectral density function estimated by DFT is given by

$$\hat{s}(\omega,t)=\frac{1}{M}\left|\sum_{k=0}^{M-1}x_k(t)e^{-j\omega k}\right|^2=\mathbf{a}^H(\omega)\mathbf{x}(t) \qquad (3)$$

where

$$\mathbf{x}(t)=\begin{bmatrix}x_0(t) & x_1(t) & \cdots & x_{M-1}(t)\end{bmatrix}^T \qquad (4)$$

is a sample sequence vector,

$$\mathbf{a}(\omega)=\frac{1}{\sqrt{M}}\begin{bmatrix}1 & e^{j\omega k} & \cdots & e^{j(M-1)\omega}\end{bmatrix}^T \qquad (5)$$

On the other hand, eigenvalue decomposition of the sample correlation matrix of a random process is defined as

$$\hat{\mathbf{R}}=\sum_{k=1}^{M}\lambda_k\mathbf{q}_k\mathbf{q}_k^H \qquad (6)$$

where λ_k is the k-th eigenvalue in descending order, $\mathbf{q}_k$ is the corresponding eigenvector, and

$$\hat{\mathbf{R}}=\sum_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^H(t) \qquad (7)$$

is the sample correlation matrix.

Next, we will discuss the relationship between Discrete Fourier Transformation and eigenvalue decomposition of autocorrelation matrix, which is defined as

$$\mathbf{R}=E\left(\mathbf{x}(t)\mathbf{x}^H(t)\right) \qquad (8)$$

B. Relationship between Definitions

As we know, there is a famous relationship between Discrete Fourier Transformation and eigenvalue decomposition of circular matrix. Circular matrix is defined as

$$\mathbf{C}=\begin{bmatrix}c_0 & c_1 & \cdots & c_{M-1}\\ c_{M-1} & c_0 & \cdots & c_{M-2}\\ \vdots & \vdots & \cdots & \vdots\\ c_1 & c_2 & \cdots & c_0\end{bmatrix} \qquad (9)$$

It is obvious that for $k=1,2,\ldots,M$, we have [5]

$$\mathbf{C}\mathbf{a}(\frac{k}{M})=\lambda_k\mathbf{a}(\frac{k}{M}) \qquad (10)$$

where

$$\lambda_k=\sum_{m=0}^{M-1}c_k e^{j\frac{k}{M}m} \qquad (11)$$

Therefore, vector given in (5) is eigenvector of circular matrix and the corresponding eigenvalue is Discrete Fourier Transformation of circular matrix elements.

Since Discrete Fourier Transformation and eigenvalue decomposition of autocorrelation matrix are used to describe the characteristics of the same random process, although from different viewpoint, there may be some intrinsic relationship between them. For example, we have

$$S_{\min}\le\lambda_k\le S_{\max} \qquad (12)$$

where λ_k is the eigenvalue of the autocorrelation matrix, $k=1,2,\ldots,M$, $S_{\min}$ and $S_{\max}$ is respectively the minimum and maximum value of the power spectral density of the same stationary random process.

Students who first get in touch with this inequality often have no way to start and feel that it is difficult to understand intuitively. However, from the point of view of signal processing, the derivation of the above inequality has a simple and clear signal processing significance.

First, the eigenvalue of the autocorrelation matrix is defined by [4]

$$\lambda_k=\frac{\mathbf{q}^H\mathbf{R}\mathbf{q}}{\mathbf{q}^H\mathbf{q}} \qquad (13)$$

where $\mathbf{R}$ and $\mathbf{q}$ is respectively the autocorrelation matrix and the eigenvector. The denominator is obviously nonnegative, and the nonnegative-definite property of autocorrelation matrix also shows the numerator is also nonnegative.

In fact, the numerator can be regarded as $r_x(0)$, i.e., the variance of a stochastic process $x(t)$, which is the output of a linear time invariant filter whose pulse response is given by $\mathbf{q}$, and the input is a stochastic process $u(n)$, whose autocorrelation matrix given as $\mathbf{R}$. Similarly, the denominator can also be regarded as $r_y(0)$, i.e., the variance of a stochastic process $y(t)$, which is the output of the same linear time invariant filter whose impulse response is given by $\mathbf{q}$, but the input is white noise process with unit variance. Therefore, we have

$$\frac{\mathbf{q}^H\mathbf{R}\mathbf{q}}{\mathbf{q}^H\mathbf{q}}=\frac{r_x(0)}{r_y(0)} \qquad (14)$$

Then, according to the relationship between the variance of a random process and the power spectrum density function [2], we have

$$\frac{r_x(0)}{r_y(0)}=\frac{\int_{-\pi}^{\pi}S_x(\omega)d\omega}{\int_{-\pi}^{\pi}S_y(\omega)d\omega} \qquad (15)$$

where $S_x(\omega)$ and $S_y(\omega)$ is the power spectral density function of the random process $x(n)$ and $y(n)$, respectively.

Next, according to the relationship between power spectral density function of the input and output stochastic process of a linear time invariant filter, we have [2]

$$\frac{\int_{-\pi}^{\pi}S_x(\omega)d\omega}{\int_{-\pi}^{\pi}S_y(\omega)d\omega}=\frac{\int_{-\pi}^{\pi}S_u(\omega)Q(\omega)d\omega}{\int_{-\pi}^{\pi}Q(\omega)d\omega} \qquad (16)$$

where $S_u(\omega)$ is he power spectral density function of the stochastic process $u(t)$, $Q(\omega)$ is the amplitude square of the frequency response of the linear time invariant system whose impulse response is $\mathbf{q}$, i.e.,

$$Q(\omega) = \left| \sum_{k=0}^{M-1} q_k e^{-j\omega k} \right|^2 \tag{17}$$

where

$$\mathbf{q} = \begin{bmatrix} q_0 & q_1 & \cdots & q_{M-1} \end{bmatrix}^T \tag{18}$$

Thus we can directly obtain

$$\lambda_k = \frac{\int_{-\pi}^{\pi} S_u(\omega) Q(\omega) d\omega}{\int_{-\pi}^{\pi} Q(\omega) d\omega} \le \frac{\int_{-\pi}^{\pi} S_{\max} Q(\omega) d\omega}{\int_{-\pi}^{\pi} Q(\omega) d\omega} = S_{\max} \tag{19}$$

and

$$\lambda_k = \frac{\int_{-\pi}^{\pi} S_u(\omega) Q(\omega) d\omega}{\int_{-\pi}^{\pi} Q(\omega) d\omega} \ge \frac{\int_{-\pi}^{\pi} S_{\min} Q(\omega) d\omega}{\int_{-\pi}^{\pi} Q(\omega) d\omega} = S_{\min} \tag{20}$$

In this section, we introduce the intrinsic relationship behind the definition of Discrete Fourier Transformation of a stochastic process and eigenvalue decomposition of its autocorrelation matrix, from which the students can grasp their related property and review the following knowledge points: (1) the relationship between the variance of a random process and the power spectrum density function. (2) The relationship between power spectral density function of the input and output stochastic process of a linear time invariant filter.

III. Relationship between the Problems Formulation

A. Problem Formulation of DFT

Let us approximate a sample sequence vector $\mathbf{x}(t)$ by using vector $\mathbf{a}(\omega)$. This problem can be formulated as follows

$$\min_{\substack{s(\omega,t)\\ t=1,2,\ldots,N}} \sum_{t=1}^{N} \left\| \mathbf{x}(t) - s(\omega,t)\mathbf{a}(\omega) \right\|_F^2 \tag{21}$$

The students may wonder why we use $\mathbf{a}(\omega)$ to approximate a sample sequence vector. The reason is that

$$v(t) = e^{j\omega t} \tag{22}$$

is 1-dimensional signal, i.e., $v(t) = e^{j\omega} v(t-1)$. In other word, we want to describe or analyze a stochastic process by the simplest signal. It can be regarded as a motivation of Discrete Fourier Transformation.

B. Problem Formulation of Eigenvalue Decomposition of Sample Correlation Matrix

Similarly, let us approximate a sample sequence vector $\mathbf{x}(t)$ by using a normalized vector $\mathbf{q}$, which is independent of time. This problem can be formulated as follows

$$\begin{aligned} &\min_{\substack{\mathbf{q},\alpha(t)\\ t=1,2,\ldots,N}} \sum_{t=1}^{N} \left\| \mathbf{x}(t) - \alpha(t)\mathbf{q} \right\|_F^2 \\ &s.t. \quad \mathbf{q}^H \mathbf{q} = 1 \end{aligned} \tag{23}$$

Also, the students may wonder why we use a normalized vector $\mathbf{q}$ to approximate a sample sequence vector. The reason is that $\mathbf{q}$ represents 1-dimensional subspace in the space where sample sequence vector $\mathbf{x}(t)$ locates. In other word, we want to describe or analyze a stochastic process in the lowest dimensional subspace. It can be regarded as one of motivations of eigenvalue decomposition of sample autocorrelation matrix.

IV. Relationship between the Solutions

A. Solution to Problem (21)

It is easy to see that the solution to least squares problem (21) is

$$\hat{s}(\omega,t) = \mathbf{a}^H(\omega)\mathbf{x}(t) = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} x_k(t) e^{-j\omega k} \tag{24}$$

for $t = 1, 2, \ldots, N$.

B. Solution to Problem (23)

It is also easy to see that the optimal solution to problem (23) is

$$\hat{\alpha}(t) = \mathbf{q}^H \mathbf{x}(t) \tag{25}$$

Substituting it into (23)yield

$$\begin{aligned} &\min_{\mathbf{q}} \sum_{t=1}^{N} \left\| \mathbf{x}(t) - \mathbf{q}\mathbf{q}^H \mathbf{x}(t) \right\|_F^2 \\ &s.t. \quad \mathbf{q}^H \mathbf{q} = 1 \end{aligned} \tag{26}$$

It is equivalent to

$$\begin{aligned} &\min_{\mathbf{q}} \operatorname{Tr}\left(\left(\mathbf{I} - \mathbf{q}\mathbf{q}^H \right) \hat{\mathbf{R}} \left(\mathbf{I} - \mathbf{q}\mathbf{q}^H \right) \right) \\ &s.t. \quad \mathbf{q}^H \mathbf{q} = 1 \end{aligned} \tag{27}$$

where Tr is the trace of a matrix and $\hat{\mathbf{R}}$ is sample correlation matrix. Since we have

$$\begin{aligned} &\operatorname{Tr}\left(\left(\mathbf{I} - \mathbf{q}\mathbf{q}^H \right) \hat{\mathbf{R}} \left(\mathbf{I} - \mathbf{q}\mathbf{q}^H \right) \right) \\ &= \operatorname{Tr}\left(\hat{\mathbf{R}} \left(\mathbf{I} - \mathbf{q}\mathbf{q}^H \right) \left(\mathbf{I} - \mathbf{q}\mathbf{q}^H \right) \right) \\ &= \operatorname{Tr}\left(\hat{\mathbf{R}} \left(\mathbf{I} - \mathbf{q}\mathbf{q}^H \right) \right) \\ &= \operatorname{Tr}\left(\hat{\mathbf{R}} - \hat{\mathbf{R}}\mathbf{q}\mathbf{q}^H \right) \\ &= \operatorname{Tr}\left(\hat{\mathbf{R}} \right) - \mathbf{q}^H \hat{\mathbf{R}} \mathbf{q} \end{aligned} \tag{28}$$

problem (27) is straightforwardly reduced to

$$\begin{aligned} &\max_{\mathbf{q}} \mathbf{q}^H \hat{\mathbf{R}} \mathbf{q} \\ &s.t. \quad \mathbf{q}^H \mathbf{q} = 1 \end{aligned} \tag{29}$$

Therefore, the solution is the eigenvector corresponding to the maximum eigenvalue of sample correlation matrix $\hat{\mathbf{R}}$, i.e.,

$$\hat{\mathbf{R}}\mathbf{q} = \lambda_{\max}\mathbf{q}^H \tag{30}$$

Substituting the above equation into (27), we attain the minimum value as

$$\min_{\substack{\mathbf{q},\alpha(t)\\ t=1,2,\ldots,N}} \sum_{t=1}^{N} \left\| \mathbf{x}(t) - \alpha(t)\mathbf{q} \right\|_F^2 = \mathrm{Tr}\left(\hat{\mathbf{R}}\right) - \lambda_{\max} = \sum_{k=2}^{M} \lambda_k \tag{31}$$

In this section, we introduced respectively the motivation to analyzing signal and information by using Discrete Fourier Transformation of sample sequence vector and eigenvalue decomposition of sample autocorrelation matrix. In order to make the conclusions more rigorous, some derivations are provided. As the result, students could have a deeper understanding on their simplicity, optimality and the reason why Discrete Fourier Transformation of sample sequence vector and eigenvalue decomposition of sample autocorrelation matrix are so popular in the field of signal and information processing.

V. Conclusion

When we analyze a signal or random process by using 1-dimensional signal, we encounter the Discrete Fourier transformation of sample sequence, whereas by using 1-dimensional subspace, we encounter eigenvalue decomposition of sample correlation matrix. Though they are two of important tools dealing with the signal or random process from different viewpoint, there are some intrinsic relationship between them. Therefore, they should not be introduced, explained or learned independently in the course of signal and information processing. The intrinsic relationship between discrete Fourier transformation of sample sequence and eigenvalue decomposition of sample correlation matrix includes the definition, problems formulation and their solutions. The results of these lecture notes can help students deepen the understanding of their characteristics on simplicity, optimality and the reason why they are so popular and why we analyze and deal with signal and information processing by using discrete Fourier transformation of sample sequence and eigenvalue decomposition of sample correlation matrix.

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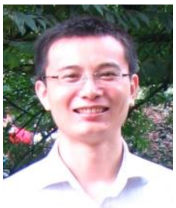

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